WEYL-PARALLEL FORMS AND CONFORMAL PRODUCTS

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Abstract. Motivated by the study of Weyl structures on conformal manifolds admitting parallel weightless forms, we define the notion of conformal product of conformal structures and study its basic properties. We obtain a classification of Weyl manifolds carrying parallel forms, and we use it to describe in further detail the holonomy of the adapted Weyl connection on conformal products.

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1. Introduction

A *conformal structure* on a smooth manifold M is an equivalence class c of Riemannian metrics modulo conformal rescalings, or, equivalently, a positive definite symmetric bilinear tensor with values in the square of the weight bundle L of M. In contrast to the Riemannian situation, there is no canonical connection on a conformal manifold. Instead of the Levi-Civita connection, one can nevertheless consider the affine space of torsion-free connections preserving the conformal structure, called *Weyl structures*.

The fundamental theorem of conformal geometry states that this space is in one-to-one correspondence with the space of connections on the weight bundle L, and is thus modeled on the vector space of smooth 1-forms. It is worth noting that not every Weyl structure is (locally) the Levi-Civita connection of a Riemannian metric in the conformal class. This actually happens if and only if the corresponding connection on L has vanishing curvature, in which case the Weyl structure is called *closed*. Every conformal problem involving closed Weyl structures is of Riemannian nature, so we will be mainly concerned with the case of non-closed Weyl structures.

Spin conformal manifolds with Weyl structures D carrying parallel spinors have been studied in [6]. The basic idea, which allows the reduction of the problem to the Riemannian case, is that the curvature tensor of a non-closed Weyl structure is no longer symmetric by pairs. This fact eventually shows that the spin holonomy representation of a non-closed Weyl structure has no fixed points, except in dimension 4, where genuine local examples do actually exist.

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We consider here the analogous question for exterior forms: Characterize (locally) those conformal manifolds (M, c) which carry an exterior form ω parallel with respect to some Weyl structure D. If D is closed, it is (locally) the Levi-Civita connection of some metric $g \in c$ and D-parallel forms correspond to fixed points of the Riemannian holonomy representation on the exterior bundle. By the de Rham theorem, and the fact that the space of fixed points of a tensor product representation is just the tensor product of the corresponding spaces of each factor, one may assume that the holonomy acts irreducibly on TM. In this case, the Berger-Simons theorem provides the list of possible holonomy groups, so the problem reduces to an algebraic (although far from being trivial) computation.

Back to the conformal setting, we remark that one can restrict ourselves to the case *weightless* forms since otherwise the Weyl structure would be automatically closed. By choosing a Riemannian metric $g \in c$, the equation $D\omega = 0$ becomes

$$\nabla^g_X \omega = \theta \wedge X \lrcorner\, \omega - X^\flat \wedge \theta^\sharp \lrcorner\, \omega \qquad \forall\ X \in TM, \tag{1}$$

where ∇^g is the Levi-Civita covariant derivative of g and θ is the connection form of D in the trivialization of L determined by g. Exterior forms satisfying (1) are called *locally conformal parallel forms* in [3] and are shown to define, under some further conditions, harmonic sections of the corresponding sphere bundles.

We start by remarking that a nowhere vanishing exterior p-form ω ($0 < p < \dim(M)$) can not be parallel with respect to more than one Weyl structure. In fact ω defines a unique "minimal" Weyl structure D^ω which is the only possible candidate for having $D\omega = 0$. We next apply the Schwachhöfer-Merkulov classification of torsion-free connections with irreducible holonomy [5] to the Weyl structure D^ω. A quick analysis of their tables shows that the possible (non-generic) holonomy groups of irreducible Weyl structures are all compact (except in dimension 4, where the solutions to our problem turn out to correspond to Hermitian structures – see Lemma 5.6). But, of course, a Weyl structure with compact (reduced) holonomy is closed since its holonomy bundle defines (local) Riemannian metrics which are tautologically D-parallel.

It remains to study the reducible case, which, unlike in the Riemannian situation, is more involved. First of all, we extend the de Rham theorem to the conformal setting. To do this, we need to define the notion of *conformal products*. Indeed, in contrast to Riemannian geometry, there is no canonical conformal structure on a product $M_1 \times M_2$ of two conformal manifolds $(M_1^{n_1}, c_1)$ and $(M_2^{n_2}, c_2)$ induced by the two conformal structures alone. The algebraic reason is, of course, that the group $\mathrm{CO}(n_1) \times \mathrm{CO}(n_2) \subset \mathrm{GL}(n_1 + n_2, \mathbb{R})$ is not included in $\mathrm{CO}(n_1 + n_2)$.

On the other hand, a property characterizing the Riemannian product (M, g) of two Riemannian manifolds (M_1, g_1) and (M_2, g_2) is the existence of two *complementary orthogonal* Riemannian submersions $p_i : (M, g) \to (M_i, g_i)$ (here, *complementary* means that TM is the direct sum of the kernels of dp_i and *orthogonal* means that these kernels are orthogonal at each point). Generalizing this to conformal geometry, a conformal structure on the manifold $M := M_1 \times M_2$ is said to be *a conformal product* of (M_1, c_1)

and (M_2, c_2) if the canonical submersions $p_1 : M \to M_1$ and $p_2 : M \to M_2$ are *orthogonal* conformal submersions.

In Section 4 we show that every conformal product carries a unique *adapted* reducible Weyl structure D preserving the two factors, and conversely, every reducible Weyl structure induces a local conformal product structure.

The similarities with the Riemannian case stop here, however, since the factors of a conformal product do not carry canonical Weyl structures (in fact the restrictions of the adapted Weyl structure D to each slice $\{x_1\} \times M_2$ or $M_1 \times \{x_2\}$ of the conformal product $M_1 \times M_2$ depends on x_1 and x_2), so it is not possible to interpret the space of D-parallel forms on $M_1 \times M_2$ in terms of the two factors.

On the other hand, the lack of symmetry of the curvature tensor of D mentioned above, allows us to show (in Section 5) that every parallel form on a conformal product with non-closed adapted Weyl structure is of pure type and eventually has to be the weightless volume form of one of the factors (except in dimension 2 or 4 where the situation is more flexible).

Note that most of the arguments and results presented here are of local nature. The compact setting is work in progress and will hopefully be described in a forthcoming paper.

2. Preliminaries

Let M^n be a manifold and let P denote the principal bundle of frames. The *weight bundle* of M is the real line bundle L associated to P via the representation $|\det|^{1/n}$ of $\mathrm{GL}(n, \mathbb{R})$. More generally one can define the k-weight bundle L^k for every $k \in \mathbb{R}$, associated to P via the representation $|\det|^{k/n}$. Obviously $L^k \otimes L^p \cong L^{k+p}$ and $L^{-n} \cong |\Lambda^n(T^*M)| = \delta M$, which is the bundle of *densities* on M, a trivial line bundle (associated to P via the representation $|\det|^{-1}$) even if M is not orientable. Positive densities are geometrically meaningful as "absolute values" of volume forms and positive global densities induce Lebesgue-like measures on M (like the well-known *Riemannian volume element* of a Riemannian manifold).

As the weight bundles are powers of δM, the notion of positivity is still well-defined; more precisely, a section of L^k is positive if it takes values in $P \times_{|\det|^{k/n}} \mathbb{R}^+ \subset L^k$. A *weighted tensor* on M is a section of $TM^{\otimes a} \otimes T^*M^{\otimes b} \otimes L^k$ for some $a, b \in \mathbb{N}$ and $k \in \mathbb{R}$. Its weight is by definition the real number $a - b + k$.

Definition 2.1. *A* conformal structure *on M is a symmetric positive definite bilinear form c on $TM \otimes L^{-1}$, or, equivalently, a symmetric positive definite bilinear form on TM with values in L^2.*

A conformal structure on M can also be seen as a reduction $P(\mathrm{CO}_n)$ of P to the conformal group $\mathrm{CO}_n \cong \mathbb{R}^+ \times \mathrm{O}_n \subset \mathrm{GL}(n, \mathbb{R})$.

We denote by $\Lambda_0^k M := \Lambda^k M \otimes L^k$ the bundle of weightless exterior forms of degree k. The conformal structure defines an isomorphism between weightless vectors and 1-forms: $TM \otimes L^{-1} \cong \Lambda_0^1 M$. The scalar product c on $\Lambda^1 M \otimes L$ induces a scalar product, also denoted by c, on the bundles $\Lambda_0^k M$. Moreover, the exterior product maps $\Lambda_0^k M \otimes \Lambda_0^{n-k} M$ onto $\Lambda_0^n M \cong \mathbb{R}$, thus defining the Hodge operator $* : \Lambda_0^k M \to \Lambda_0^{n-k} M$ by

$$\omega \wedge \sigma = c(\omega, \sigma), \qquad \forall \omega, \sigma \in \Lambda_0^k M. \tag{2}$$

There is a one-to-one correspondence between positive sections l of L and Riemannian metrics on M, given by the formula

$$c(X, Y) = g(X, Y) l^2, \qquad \forall\ X, Y \in TM. \tag{3}$$

Definition 2.2. *A* Weyl structure *on a conformal manifold is a torsion-free connection on $P(\mathrm{CO}_n)$.*

By (2), the Hodge operator is parallel with respect to every Weyl structure.

Theorem 2.3. (Fundamental theorem of Weyl geometry) *There is a one-to-one correspondence between Weyl structures and covariant derivatives on L.*

Proof. Every connection on P induces a covariant derivative D on TM and on L. The connection is a Weyl structure if and only if D satisfies $D_X Y - D_Y X = [X, Y]$ and $D_X c = 0$ for all vector fields $X,\ Y$ on M. Like in the Riemannian situation, these two relations are equivalent to the Koszul formula

$$\begin{aligned} 2c(D_X Y, Z) =& D_X(c(Y, Z)) + D_Y(c(X, Z)) - D_Z(c(X, Y)) \\ &+ c([X, Y], Z) + c([Z, X], Y) + c([Z, Y], X), \end{aligned} \tag{4}$$

for all vector fields $X,\ Y,\ Z$ on M. We thus see that every covariant derivative D on L induces by the formula above a covariant derivative on TM, and thus on P, which is clearly torsion-free and satisfies $Dc = 0$. □

A Weyl structure D is called *closed* (resp. *exact*) if L carries a local (resp. global) D-parallel section. As Riemannian metrics in the conformal class c correspond to positive sections of L, it follows immediately that D is closed (resp. exact) if and only if D is locally (resp. globally) the Levi-Civita connection of a metric $g \in c$.

If D and D' are covariant derivatives on L, their difference is determined by a 1-form τ: $D_X l - D'_X l = \tau(X) l$ for all $X \in TM$ and sections l of L. From (4) we easily obtain

$$D_X Y - D'_X Y = \tau(X) Y + \tau(Y) X - c(X, Y)\tau,$$

for all vector fields X, Y. Here we note that the last term on the right hand side, which is a section of $L^2 \otimes \Lambda^1 M$, is identified with a vector field using the conformal structure.

For every $X \in TM$ we define the endomorphism $\tilde\tau_X$ on TM and on L by

$$\tilde\tau_X(Y) := \tau(X) Y + \tau(Y) X - c(X, Y)\tau, \qquad \tilde\tau_X(l) := \tau(X) l, \tag{5}$$

and extend it as a derivation to all weighted tensor bundles (in particular $\tilde{\tau}_X$ is the scalar multiplication by $k\tau(X)$ on L^k). We then have

$$D_X - D'_X = \tilde{\tau}_X, \tag{6}$$

on all weighted bundles.

Consider now a metric g in the conformal class c, or equivalently, a positive section l of L trivializing L. Let D be a Weyl structure on (M, c) and let $\theta \in \Omega^1(M, \mathbb{R})$ be the connection form of D on L with respect to the gauge l:

$$D_X l = \theta(X) l, \qquad \forall\ X \in TM. \tag{7}$$

The 1-form θ is called the *Lee form* of D with respect to g. The curvature of D on L is the two-form $F := d\theta$ called the *Faraday form*.

Let R^D denote the curvature tensor of a Weyl structure D, defined as usual for vector fields X, Y and Z by $R^D_{X,Y}Z = [D_X, D_Y]Z - D_{[X,Y]}Z$. We also view R^D as a section of $T^*M^{\otimes 4} \otimes L^2$ by the formula $R^D(X,Y,Z,T) = c(R^D_{X,Y}Z, T)$. In contrast to the Riemannian case, R^D is not symmetric by pairs, and a straightforward calculation shows that the symmetry failure is measured by the Faraday form F of D:

$$\begin{aligned} R^D(X,Y,Z,T) - R^D(Z,T,X,Y) \ &= \ (F(X) \wedge Y - F(Y) \wedge X)(Z,T) \\ &\quad + F(X,Y)c(Z,T) - F(Z,T)c(X,Y), \end{aligned} \tag{8}$$

where, for a (weighted) endomorphism $A \in \mathrm{End}(TM) \otimes L^k$, and vectors X, Y, Z, T:

$$(A(X) \wedge Y)(Z,T) := c(A(X), Z)c(Y,T) - c(A(X), T)c(Y,Z).$$

3. Conformal submersions

If (M^m, c) and (N^n, c') are conformal manifolds, a *conformal map* is a smooth map $f : M \to N$ such that

$$df|_{(\ker df)^\perp} : (\ker df)^\perp \to df(T_xM) \subset T_{f(x)}N$$

is a conformal isomorphism for every $x \in M$. A conformal map which is a submersion is called a conformal submersion.

Lemma 3.1. *Let (M^m, c) be a conformal manifold and let $p : M \to N$ be a submersion onto a manifold N^n. Then the pull-back of the weight bundle of N is canonically isomorphic to the weight bundle of M.*

Proof. Let L' and L denote the weight bundles of N and M respectively. We decompose $TM = \ker dp \oplus (\ker dp)^\perp$ into the vertical and horizontal distributions. We will show that $p^*(L')^{-n}$ is canonically isomorphic to L^{-n}. Every element $(l')^{-n}$ of the fiber of $(L')^{-n} \simeq \delta N$ at $y \in N$ can be represented by the density $(l')^{-n} := |\varepsilon_1 \wedge ... \wedge \varepsilon_n|$ where $\{\varepsilon_i\}$ is some basis of T^*N_y. For every $x \in p^{-1}(y)$ we then associate to the element $p^*(l')^{-n}_y$ the conformal norm of $(p^*\varepsilon_1)_x \wedge ... \wedge (p^*\varepsilon_n)_x$, which is an element of L^{-n}_x. It is straightforward to check that this isomorphism does not depend on the choice of the basis.

□

Remark 3.2. Notice that this result only holds for $n \geq 1$ since we need at least one non-vanishing 1-form in order to produce a weight on N.

Lemma 3.3. *Let $p : M \to N$ be a submersion with connected fibers from a conformal manifold (M^m, c) onto a manifold N^n. Assume that the horizontal distribution $H := (\ker df)^\perp$ is parallel with respect to some Weyl structure D. Then the pull-back to M of every covariant weighted tensor on N is D-parallel in the vertical directions.*

Conversely, a covariant weighted tensor on M which is horizontal and D-parallel in the vertical directions, is the pull-back of a covariant weighted tensor on N.

Proof. We first show that $D_V(p^*\omega) = 0$ for all 1-forms ω on N. If W is a vertical vector field, $D_V W$ is again vertical, so $0 = p^*\omega(D_V W) = (D_V(p^*\omega))(W)$. Next, if X is another vector field on N and $\tilde{X}$ denotes its horizontal lift, $p^*\omega(\tilde{X})$ is constant on each fiber of p, so $0 = V.(p^*\omega(\tilde{X})) = (D_V(p^*\omega))(\tilde{X}) + (p^*\omega)(D_V\tilde{X})$. On the other hand $D_V\tilde{X} = D_{\tilde{X}}V + [V, \tilde{X}]$ vanishes (because $D_{\tilde{X}}V$ and $[V, \tilde{X}]$ are vertical, and $D_V\tilde{X}$ is horizontal), so finally $D_V(p^*\omega) = 0$.

Lemma 3.1 shows that the pull-back of the weight bundle L' of N is isomorphic to the weight bundle L of M. Moreover, the calculation above shows that

$$D_V(p^*l') = 0 \tag{9}$$

for every section l' of L' and vertical vector field $V \in \ker df$. Since the 1-forms and the sections of L' generate the whole algebra of covariant weighted tensors on N, this proves the first part of the lemma.

For the converse part, we first show that a section l of L which is parallel in vertical directions is the pull-back of a section of $L' \to N$. Indeed, if we take any global nowhere vanishing section l' of L', one can write $l = fp^*l'$ for some function f which by (9) is constant in vertical directions, i.e. f is the pull-back of some function f' on N, so finally $l = p^*(f'l')$.

Let now $Q : TM^{\otimes k} \to L^r$ be a covariant weighted tensor field on M such that $Q(X_1, \ldots, X_k) = 0$ whenever one of the X_i is vertical, and which is D-parallel in the vertical directions. For every $y \in N$ and $x \in p^{-1}(y) \subset M$ we define

$$\bar{Q}_y(Y_1, \ldots, Y_k) := Q_x(\tilde{Y}_1, \ldots, \tilde{Y}_k), \tag{10}$$

where $Y_1, \ldots, Y_k \in T_yN$ are arbitrary vectors, and $\tilde{Y}_1, \ldots, \tilde{Y}_k \in T_xM$ are their horizontal lifts. This definition makes sense thanks to the identification of $p^*L' \simeq L$ of Lemma 3.1. In order to show that it is independent of the choice of $x \in p^{-1}(y)$, we need to check that the right hand side is a weight which is D-parallel in vertical directions. This follows from the relations $D_V\tilde{Y}_i = 0$, proved above, and the hypothesis $D_VQ = 0$.

□

Corollary 3.4. *Under the hypothesis of the previous lemma, there exists a unique conformal structure on N turning p into a conformal submersion.*

Proof. Let c_1 denote the restriction of the conformal structure on M to the horizontal distribution. Since the horizontal distribution is D-parallel, the same holds for c_1. Lemma 3.3 thus shows that c_1 is the pull-back of some weighted tensor c' on N, which is clearly a conformal structure on N.

□

Remark 3.5. We can extend now the result of the Lemma 3.3 to *any* weighted tensor on N, respectively on M, because on a conformal manifold every tensor can be seen as a covariant one (with the appropriate weight).

4. Conformal products

Let (M_1, c_1) and (M_2, c_2) be two conformal manifolds, $M = M_1 \times M_2$ and let $p_i : M \to M_i$ be the canonical submersions.

Definition 4.1. *A conformal structure on the manifold $M := M_1 \times M_2$ is said to be* a conformal product *of (M_1, c_1) and (M_2, c_2) if and only if the canonical submersions $p_1 : M \to M_1$ and $p_2 : M \to M_2$ are* orthogonal *conformal submersions.*

For later use, we describe the construction of a conformal product structure in terms of weight bundles:

Proposition 4.2. *Given two conformal manifolds $(M_1^{n_1}, c_1)$, resp. $(M_2^{n_2}, c_2)$, there is a one-to-one correspondence between the set of* conformal product *structures on $M := M_1 \times M_2$ and the set of pairs of bundle homomorphisms $P_1 : L \to L_1$ and $P_2 : L \to L_2$, whose restrictions to each fiber are isomorphisms, such that the following diagram is commutative (here L, L_1, L_2 denote the weight bundles of M, M_1, and M_2 respectively):*

$$\begin{tikzcd} & & L \arrow[lldd, "P_1"'] \arrow[d] \arrow[rrdd, "P_2"] & & \\ & & M \arrow[ld, "p_1"'] \arrow[rd, "p_2"] & & \\ L_1 \arrow[r] & M_1 & & M_2 & L_2 \arrow[l] \end{tikzcd} \tag{11}$$

Proof. It is a general fact that for a *conformal map* $f : (M, c) \to (N, c')$ between two conformal manifolds, there is a canonically associated bundle map $f^L : L_M \to L_N$ of the weight bundles, isomorphic on each fiber (take any non-zero vector X in $(\ker df)^\perp$ and define $f^L(\sqrt{c(X,X)}) = \sqrt{c'(f_*X, f_*X)}$).

Therefore, given a conformal product structure on $M \simeq M_1 \times M_2$ (i.e., a pair of conformal submersions $p_1 : M \to M_1$, resp. $p_2 : M \to M_2$), we associate to it the induced bundle homomorphisms $P_i := p_i^L$, such that the diagram (11) commutes.

Conversely, let $M = M_1 \times M_2$ and let L denote the *weight bundle* of the product manifold M (which does not have any conformal structure yet). Let $P_i : L \to L_i$ be line bundle homomorphisms making the diagram (11) commutative. The condition that P_i are isomorphic on each fiber just means that the pull-back bundles $p_i^* L_i$ are both isomorphic with L. We then define $\tilde{c}_i \in Sym^2(M) \otimes p_i^*(L_i^2) \cong Sym^2(M) \otimes L^2$ by $\tilde{c}_i(X,Y) := p_i^*(c_i((p_i)_*(X),(p_i)_*(Y)))$ and $c = \tilde{c}_1 + \tilde{c}_2$.

□

The following theorem, which is the main result of this section, establishes the existence of a unique adapted Weyl structure on a conformal product:

Theorem 4.3. *A conformal structure c on a manifold M is a (local) conformal product structure if and only if it carries a Weyl structure with reducible holonomy. Moreover, the correspondence between the conformal product structures and the Weyl structures with reduced holonomy is one-to-one.*

Proof. Let us first prove that if a conformal manifold (M,c) carries a Weyl structure D with reduced holonomy, then it is locally a conformal product. Let $TM = H_1 \oplus H_2$ be a D-invariant splitting of the tangent bundle. Because D is torsion-free, the distributions H_1 and H_2 are integrable, therefore we have two orthogonal foliations on M tangent to these distributions. Locally, M is then a product manifold $M_1 \times M_2$, and the foliations above are the fibers of the canonical projections $p_i : M \to M_i$. Corollary 3.4 then shows that there exist conformal structures c_1, c_2 on M_1, resp. M_2, such that the canonical projections are conformal submersions.

Conversely, suppose (M,c) is the a conformal product with factors (M_1,c_1), resp (M_2,c_2). We look for a Weyl structure D that preserves the canonical splitting $TM = H_1 \oplus H_2$, with $H_1 := \ker dp_2$ and $H_2 := \ker dp_1$.

By Theorem 2.2, the set of Weyl structures is in 1–1 correspondence with the set of connections on the weight bundle. Therefore, it is enough to specify the corresponding connection D on L. We describe D using the diagram (11) as follows: the horizontal space H_l of D at $l \in L$ is the direct sum of $\ker dP_1$ and $\ker dP_2$. This horizontal space defines a linear connection because the maps P_1, P_2 commute with the scalar multiplication on the fibers. In terms of covariant derivative, this definition amounts to say that the pull-back of a section of L_1 (resp. L_2) is D-parallel in the direction of H_2 (resp. H_1). We need to show that the induced Weyl structure D preserves H_1 and H_2.

Since the rôles of H_1 and H_2 are symmetric, it is enough to prove that $c(D_X Y, Z) = 0$ for all vector fields $X, Y \in H_1$ and $Z \in H_2$. Of course, we may assume that X, Y are lifts of vector fields on M_1 and Z is a lift of a vector field on M_2. Then the brackets $[X,Z]$ and $[Y,Z]$ vanish, because the vector fields X and Y, resp. Z are defined on different factors of the product $M_1 \times M_2$, and the scalar products $g(X,Z) = g(Y,Z) = 0$ for the same reason. The only a priori non-vanishing terms in the Koszul formula (4) are thus:

$$2c(D_X Y, Z) = -D_Z(c(X,Y)) + c([X,Y],Z). \tag{12}$$

The first term vanishes by the definition of D and the second one because $[X,Y] \in H_1$ and $Z \in H_2$.

□

Definition 4.4. *The Weyl structure defined on a conformal product* (M, c) *by the result above is called the* adapted Weyl structure.

Definition 4.5. *A conformal product* (M, c) *is called a* closed conformal product *if the adapted Weyl structure is closed.*

It is easy to check that a conformal product is closed if and only if the diagram (11) can be completed by bundle homomorphisms $Q_i : L_i \to L_0$, isomorphic on each fiber (where L_0 is the weight bundle of the point manifold $\bullet$), such that the resulting diagram is commutative as well:

$$\begin{tikzcd}
 & & L \arrow[dll, "P_1"'] \arrow[d] \arrow[drr, "P_2"] & & \\
 & & M \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & & \\
L_1 \arrow[r] \arrow[ddrr, "Q_2"'] & M_1 \arrow[dr] & & M_2 \arrow[dl] & L_2 \arrow[l] \arrow[ddll, "Q_1"] \\
 & & \bullet & & \\
 & & L_0 \arrow[u] & &
\end{tikzcd} \tag{13}$$

Lemma 4.6. *Let F be the Faraday form of the adapted Weyl structure on a conformal product. Then $F(X,Y) = 0$ if $X, Y \in H_1$ or $X, Y \in H_2$.*

Proof. Let l_1 be a section of the weight bundle L_1 of M_1. Lemma 3.1 shows that its pull-back can be identified with a section l of L, and Lemma 3.3 shows that $D_X l = 0$ for every $X \in H_2$. Let X, Y be sections of H_2. Since H_2 is involutive, we have $[X,Y] \in H_2$, so

$$F(X,Y)l = D_X D_Y l - D_Y D_X l - D_{[X,Y]} l = 0.$$

The vanishing of F on H_1 is similar.

□

Remark 4.7. One can show that, for any given distribution E on a conformal manifold M, there is a unique *adapted* Weyl structure ∇, in the sense that some naturally defined tensors, depending on the splitting of $TM \simeq E \oplus E^\perp$ have minimal covariant derivative (see [2], Prop. 3.3).

Lemma 4.8. *Let (M_1, c_1) and (M_2, c_2) be two conformal manifolds and let c be a conformal product structure on $M = M_1 \times M_2$ with adapted Weyl structure D. Then each* *slice $M_1 \times \{y\} \simeq M_1$ carries a Weyl structure D^y such that $p_1^*(D^y_X T) = D_X(p_1^* T)$ at points of the form (x, y) for all vectors X and tensor fields T on M_1. The Weyl structure D is closed if and only if all connections D^y coincide.*

Proof. The restriction to each slice $M_1 \times \{y\}$ defines a covariant derivative D^y on M_1, which by Lemmas 3.1 and 3.3 preserves the conformal structure of M_1. In order to prove the last statement, it is enough to consider the case when T is a vector field on M_1. We consider vector fields U_2 on M_2 and $X_1,\ Y_1,\ Z_1$ on M_1, and denote their canonical lifts to M by U, respectively by X, Y, Z. Lemma 3.3 shows that $D_U Y = 0$, which together with $[U, X] = 0$ yields $R^D(U, X, Y, Z) = c(D_U D_X Y, Z)$. On the other hand, $R^D_{Y,Z} U$ is tangent to M_2, so $R^D(Y, Z, U, X)$ vanishes. Plugging these two relations into (8) yields

$$c(D_U D_X Y, Z) = (Z \wedge F(Y) - Y \wedge F(Z))(U, X) + F(U, X)c(Y, Z). \tag{14}$$

If D is closed, (i.e. $F = 0$) we thus get $D_U D_X Y = 0$, which just means (by Lemma 3.3 again) that the vector field $D_X Y$ is the lift to M of a vector field on M_1, and thus $D^y_X Y$ does not depend on y.

Conversely, if D^y does not depend on y, we get $D_U D_X Y = 0$, therefore $R^D(U, X)Y = 0$. But the endomorphism $R^D(U, X) : TM \to TM$ decomposes as a skew-symmetric endomorphism (in Riemannian geometry this is the only piece) and a symmetric one, which is equal to $F(U, X)\mathrm{Id}$. This piece has to vanish now that $R^D(U, X)Y = 0$, hence $F(U, X) = 0$. This, together with Lemma 4.6, proves that $F = 0$.

□

Remark 4.9. Given two conformal manifolds (M_1, c_1) and (M_2, c_2), let us fix some Riemannian metrics $g_i \in c_i$. Every conformal product structure c on $M = M_1 \times M_2$ is of the form

$$c = [g] = [e^{f_1} g_1 + e^{f_2} g_2], \tag{15}$$

where f_i are functions on M. Moreover, for two couples (f_1, f_2), resp. (f_1', f_2'), the resulting conformal structures are equal if and only if $f_1 - f_2 = f_1' - f_2'$. One can check that the Lee form of the adapted Weyl structure of c with respect to g is given by $\theta(X) = -X_1(f_2) - X_2(f_1)$, where X_i denote the components of X with respect to the decomposition $TM = TM_1 \oplus TM_2$. The conformal class c is a closed conformal product if and only if it can be expressed in the form (15), each f_i being a pull-back of a function on M_i.

Remark 4.10. The results in this section hold under the implicit assumption that each factor of the conformal product has dimension at least one, because we need to identify the weight bundle of the product with the pull-back of those of each factor (see Remark 3.2).

5. Weyl-parallel forms

In this section we study the following problem, which motivates, as we shall see below, the notion of conformal product. Given a conformal manifold (M^n, c), and a weightless

k-form $\omega \in \Lambda_0^k M$ for some $1 \leq k \leq n-1$, does there exist a Weyl structure D such that ω is D-parallel?

Remark 5.1. We only consider weightless forms since if T is a (non-vanishing) D-parallel weighted tensor, then $T/\sqrt{c(T,T)}$ is a D-parallel weightless tensor. Moreover, if T has non-zero weight, the Weyl structure is exact (due to the fact that the conformal norm $c(T,T)$ is a D-parallel weight).

Remark 5.2. For a conformal product $M = M_1 \times M_2$, the weightless volume forms on the factors induce weightless forms which are parallel with respect to the unique adapted Weyl structure (which preserves the splitting $TM \simeq TM_1 \oplus TM_2$). This will turn out to be one of the main examples of parallel weightless forms on conformal manifolds.

We start with the following useful result.

Lemma 5.3. *Let $\omega \in \Lambda_0^k M$ be a weightless k-form ($1 \leq k \leq n-1$). Then there exists* at most *one Weyl structure with respect to which ω is parallel.*

Proof. Assume that $D\omega = D'\omega = 0$, denote $D' = D + \tilde{\tau}$ like in (6) and let $g \in c$ be any ground metric, used to identify 1-forms and vectors. For every vector field X we get

$$0 = D_X\omega - D'_X\omega = \tilde{\tau}_X(\omega) = X \wedge (\tau \lrcorner \omega) - \tau \wedge (X \lrcorner \omega). \tag{16}$$

Taking the exterior product with X, and setting X to be an element of some g-orthonormal basis $\{e_i\}$ we get, after adding up all the resulting equations, that $0 = \tau \wedge (k\omega)$. Similarly, taking the interior product with X and summing over some g-orthonormal basis $X = e_i$ yields $0 = (n-k+1)(\tau \lrcorner \omega) - \tau \lrcorner \omega$, thus $\tau \lrcorner \omega = 0$. But, if $\tau \neq 0$, the condition $\tau \wedge \omega = 0$ implies that τ is a factor of $\omega = \tau \wedge \omega'$, and $\tau \lrcorner \omega = 0$ implies $\omega' = 0$ which contradicts the non-triviality of ω.

□

Remark 5.4. Consider the linear map $\alpha : T^*M \to T^*M \otimes \Lambda_0^k M$ defined by

$$\alpha(\tau)(X) = \tilde{\tau}_X(\omega).$$

The proof of the lemma above show that if ω is a nowhere vanishing section of $\Lambda_0^k M$, then α is injective, and there exists a unique Weyl structure D^ω such that $D^\omega\omega$ is orthogonal to the image of α. We call D^ω the *minimal* Weyl structure associated to ω.

Our problem can thus be reformulated as follows: Given a conformal manifold (M^n, c), find all nowhere vanishing sections ω of $\Lambda_0^k M$ for some $1 \leq k \leq n-1$, such that $D^\omega\omega = 0$. Notice that ω being nowhere vanishing is a necessary condition for the existence of a Weyl structure D with $D\omega = 0$.

We start with the case where the minimal Weyl structure D^ω associated to ω is closed. Since our study is local, there exists some metric $g \in c$ whose Levi-Civita connection is D^ω. Since g trivializes the weight bundle, ω induces a parallel k-form on M. Using the Berger-Simons holonomy theorem, our problem in this case reduces to a purely algebraic one and its answer can be synthesized in the following classical statement.

Theorem 5.5. *Let (M^n, g) be a simply connected Riemannian manifold with Levi-Civita covariant derivative ∇. The space of parallel k-forms on M is isomorphic to the space of fixed points of the holonomy group* $\mathrm{Hol}(\nabla)$ *of ∇ acting on $\Lambda^k(\mathbb{R}^n)$. If* $\mathrm{Hol}(\nabla)$ *acts irreducibly on $\mathbb{R}^n$, then either $M = G/H$ is a symmetric space, $Hol(\nabla) = H$ and H is listed in* [1]*, Tables 1-4, pp.201-202, or $Hol(\nabla)$ belongs to the Berger list (*[1]*, Corollary 10.92). If* $\mathrm{Hol}(\nabla)$ *acts reducibly on $\mathbb{R}^n$, then* $\mathrm{Hol}(\nabla)$ *is diagonally embedded in* SO_n *as a product* $\mathrm{Hol}(\nabla) = H_1 \times \ldots \times H_l$*, where each $H_i \in \mathrm{SO}_{n_i}$ belongs to the lists above.*

Another preliminary result concerns the special case of a weightless 2-form whose associated endomorphism is an almost complex structure J. This case is also classical and completely understood (see e.g. [7], Section 2):

Lemma 5.6. *An almost complex structure J compatible with the conformal structure on (M^{2m}, c) is parallel with respect to some Weyl structure D if and only if*

- *$m = 1$ or 2 and J is integrable;*
- *$m \geq 3$ and (M, c, J) is a locally conformally Kähler manifold.*

Proof. We provide the proof for the reader's convenience. Assume that $DJ = 0$. Since D is torsion free, J is integrable ([4], Ch. 9, Corollary 3.5). Let g be any metric in the conformal class c with the associated 2-form $\omega(X,Y) := g(JX,Y)$. Using g, we identify 1-forms with vectors, and 2-forms with skew-symmetric endomorphisms. If τ denotes the Lee form of D^J with respect to the Levi-Civita connection ∇ of g, we have for every tangent vector X

$$0 = D^J_X = \nabla_X J + \tilde{\tau}_X J. \tag{17}$$

From (5) we compute

$$\begin{aligned}
(\tilde{\tau}_X J)(Y) &= \tau_X(JY) - J(\tau_X Y)\\
&= \tau(X)JY + \tau(JY)X - g(X,JY)\tau - J(\tau(X)Y + \tau(Y)X - c(X,Y)\tau)\\
&= (X \wedge J\tau + JX \wedge \tau)(Y),
\end{aligned}$$

whence

$$\nabla_X \omega = -(X \wedge J\tau + JX \wedge \tau) \tag{18}$$

From (17) we get in a local orthonormal basis $\{e_i\}$:

$$d\omega = \sum_i e_i \wedge \nabla_{e_i}\omega = -\sum_i e_i \wedge (e_i \wedge \tau + Je_i \wedge \tau) = -2\omega \wedge \tau.$$

Taking the exterior derivative in this last relation yields

$$0 = d^2\omega = -2d\omega \wedge \tau - 2\omega \wedge d\tau = -2\omega \wedge d\tau.$$

For $m \geq 3$ the exterior product with ω is injective, so $d\tau = 0$, and (M, c, J) is thus locally conformally Kähler.

Conversely, assume first that $m \geq 3$ and (M, c, J) is locally conformally Kähler. If g_1 and g_2 are local Kähler metrics on open sets U_1 and U_2 in the conformal class c, then

g_1 and g_2 are homothetic on $U_1 \cap U_2$. The Levi-Civita connections of all such metrics thus define a global Weyl structure D leaving J parallel.

If $m = 2$ and J is integrable, let g be any metric in the conformal class c with the associated 2-form $\omega(X, Y) := g(JX, Y)$. Since the wedge product with ω defines an isomorphism $\Lambda^1 M \cong \Lambda^3 M$, there exists a unique 1-form τ such that $d\omega = -2\omega \wedge \tau$. From [4], Ch. 9, Proposition 4.2 we obtain

$$\begin{aligned} \nabla_X \omega(Y, Z) &= -(\omega \wedge \tau)(X, JY, JZ) + (\omega \wedge \tau)(X, Y, Z) \\ &= -(JX \wedge \tau)(JY, JZ) + (JX \wedge \tau)(Y, Z) \\ &= (X \wedge J\tau + JX \wedge \tau)(Y, Z) \end{aligned}$$

(notice that in the definition of $d\omega$ there is an extra factor 3 with the conventions in [4]). By (18), this means that J is parallel with respect to the Weyl structure $\nabla + \tilde{\tau}$.

□

Before stating our main result, we need one more preliminary statement concerning conformal products.

Proposition 5.7. *A conformal product $M = M_1 \times M_2$, with $\dim M_i = n_i \geq 1$, admitting a non-trivial weightless form $\omega \in C^\infty\left(p_1^*(\Lambda_0^k M_1)\right)$, $1 \leq k \leq n_1 - 1$, which is parallel with respect to the adapted Weyl structure, is a closed conformal product.*

In other words, on a non-closed conformal product, the only weightless forms of *pure type* M_1 or M_2 are the volume forms of the factors.

Proof. Let $M = M_1 \times M_2$ and $TM = H_1 \oplus H_2$ be the corresponding orthogonal splitting of the tangent space $H_i \simeq p_i^* TM_i$, where $p_i : M \to M_i$ are the canonical projections. Let D be the adapted Weyl structure on M. Since $D_X \omega = 0$, $\forall\, X \in H_2$, Lemma 3.3 shows that ω is the pull-back of a weightless form $\omega_1 \in \mathbb{C}^\infty\left(\Lambda_0^k M_1\right)$, hence

$$\omega = p_1^* \omega_1.$$

From Lemma 4.8, D induces a Weyl structure D^y on each slice $M_1 \times \{y\}$ and the equation $D_Y \omega = 0$, $\forall\, Y \in H_1$, shows that the restriction of ω to each slice $M_1 \times \{y\}$ is D^y-parallel. Of course, all these restrictions coincide with ω_1, so we have

$$D_X^y \omega_1 = 0, \ \forall X \in TM_1 \text{ and } \forall y \in M_2.$$

Now, as the degree of ω_1 lies between 1 and $n_1 - 1$, the equation above implies (using Lemma 5.3) that all Weyl structures D^y coincide, so by Lemma 4.8 again, the Weyl structure D is closed and (M, c) is thus a closed conformal product.

□

We are now ready for the classification of conformal manifolds carrying conformally parallel forms.

Theorem 5.8. *Let (M^n, c) be a conformal manifold and let $\omega \in \mathcal{C}^\infty(\Lambda_0^k M)$ be a weightless k-form ($1 \leq k \leq n-1$) such that there exists a Weyl structure D with respect to which ω is parallel. Then the following (non-exclusive) possibilities occur:*

(1) *D is closed, so Theorem 5.5 applies.*
(2) *M has dimension 4, $k = 2$, the endomorphism of TM corresponding to ω is, up to a constant factor, an integrable complex structure, and D is its canonical Weyl structure.*
(3) *(M^n, c) is a conformal product of (M^{n_1}, c_1) and (M^{n_2}, c_2), D is the adapted Weyl structure, and*
   (a) *if $n_1 \neq n_2$, $\omega = \lambda\omega_i$ for some $\lambda \in \mathbb{R}$, where ω_i denotes the weightless volume form of the factor M_i;*
   (b) *if $n_1 = n_2$, then $\omega = \lambda\omega_1 + \mu\omega_2$ for some $\lambda, \mu \in \mathbb{R}$.*

Proof. Let us first consider the case of forms of low degree. If ω is a D-parallel weightless 1-form, its kernel defines a D-parallel distribution of codimension 1, so by Theorem 4.3 (M, c) is a conformal product where one factor is one-dimensional and ω is its weightless volume form (case 3a).

The case when ω has degree 2 has a special geometrical meaning, since it can be seen as a skew-symmetric endomorphism J of TM. As such, its square is a parallel symmetric endomorphism, therefore its eigenvalues are constant (and non-positive) and the corresponding eigenspaces are parallel. There are two cases to be considered.

If J^2 has only one eigenvalue, one may assume after rescaling that $J^2 = -\mathrm{id}_{TM}$, so by Lemma 5.6, either $n = 4$ and we are in case 2, or $n \geq 6$, (M, c) is locally conformally Kähler, and D is closed (case 1).

If J^2 has at least two eigenvalues, we denote by H_1 one of the eigenspaces, and by H_2 its orthogonal complement in TM. The splitting $TM = H_1 \oplus H_2$ is thus D-parallel, so M has to be a non-trivial conformal product by Theorem 4.3. On the other hand, J splits into $J = J_1 + J_2$, where J_1, J_2 are the restrictions of J to H_1, resp. H_2. The form ω splits accordingly into $\omega = \omega_1 + \omega_2$, with $\omega_i \in \mathcal{C}^\infty(p_i^*(\Lambda_0^2 M_i))$. By Proposition 5.7, either the conformal product $M_1 \times M_2$ is closed (case 1), or the non-trivial ω_i is a pull-back of a weightless volume form on M_i. Therefore, if D is non-closed, J^2 has exactly two eigenvalues, which are either both non-zero (then $n = 4$ and we are in case 3b) or only one is non-zero, and we are in case 3a. Note that in the latter case, ω is defined as the pull-back of a volume form on a 2-dimensional conformal factor, thus it is decomposable.

In order to proceed, we make use of Schwachhöfer-Merkulov's classification of torsion-free connections with irreducible holonomy [5]. Their result, in the particular case of Weyl structures, states that there are four possibilities: Either $n = 4$, or D has full holonomy $CO^+(n)$, or D is closed, or D has reducible holonomy.

If $n = 4$, the case where the degree of ω is 1 or 2 has already been considered, and if ω has degree 3, its Hodge dual is again a D-parallel weightless 1-form, so we are in case 3a.

If D has full holonomy $CO^+(n)$, there is of course no D-parallel weightless k-form on M for $1 \leq k \leq n-1$.

If D is closed we are already in case 1.

For the rest of the proof, we thus may assume that the holonomy of D acts reducibly on TM and $\dim M \geq 5$. By Theorem 4.3, (M^n, c) is locally a conformal product of (M^{n_1}, c_1) and (M^{n_2}, c_2) and D is the adapted Weyl structure. From Lemma 3.3 we have the following D-parallel decomposition:

$$\Lambda_0^k M \simeq \bigoplus_{k_1+k_2=k} p_1^*(\Lambda_0^{k_1} M_1) \otimes p_2^*(\Lambda_0^{k_2} M_2). \tag{19}$$

If for every $k_1 \geq 1,\ k_2 \geq 1$ the components of ω in $p_1^*(\Lambda_0^{k_1} M_1) \otimes p_2^*(\Lambda_0^{k_2} M_2)$ vanish, then $\omega = \omega_1 + \omega_2$, with $\omega_i \in C^\infty\left(p_i^*(\Lambda_0^k M_i)\right)$, and ω_1, ω_2 are both parallel. Proposition 5.7 then implies that either D is closed (case 1) or ω_i are both pull-backs of weightless volume forms on the factors (not both trivial). But this can only happen if k is equal to one of the dimensions n_1, n_2 (case 3a) or to both of them (case 3b).

To deal with the cases when ω is not a (combination of) pure type form, we may assume without loss of generality that ω is a non-trivial section of $p_1^*(\Lambda_0^{k_1} M_1) \otimes p_2^*(\Lambda_0^{k_2} M_2)$ with $k_1 \geq 1,\ k_2 \geq 1$. By considering the Hodge dual of ω (which is D-parallel as well), we may even assume $k_1 < n_1$ and $k_2 < n_2$. We will show that, in this case, D must be closed (case 1).

Let R^D denote the curvature tensor of D. If $X \in TM_1$ and $A \in TM_2$, we have $R^D(\cdot, \cdot, X, A) = 0$. Since $D\omega = 0$ we also have $R^D(X, A)(\omega) = 0$. From (8) we thus get

$$R^D(X, A, \cdot, \cdot) = (F(X) \wedge A - F(A) \wedge X)(\cdot, \cdot) + F(X, A)c(\cdot, \cdot),$$

and note that the first part is a skew-symmetric endomorphism of TM, and the second term is a multiple of the identity. That last one acts trivially on weightless forms, so we are left with four terms in $R^D(X, A)(\omega)$ and get:

$$F(A) \wedge (X \lrcorner\, \omega) - X \wedge (F(A) \lrcorner\, \omega) - F(X) \wedge (A \lrcorner\, \omega) + A \wedge (F(X) \lrcorner\, \omega) = 0, \tag{20}$$

for all $X \in TM_1$ and $A \in TM_2$.

We now choose a metric $g \in c$ and identify the weightless form ω with the corresponding p-form of constant g-length. If θ denotes the Lee form of D with respect to g and ∇ the Levi-Civita covariant derivative of (M, g), we have

$$\nabla_U \omega = -\tilde{\theta}_U \omega = \theta \wedge (U \lrcorner\, \omega) - U \wedge (\theta \lrcorner\, \omega). \tag{21}$$

By contraction we easily get $d\omega = -p\theta \wedge \omega$. Taking the exterior derivative in this relation yields

$$d\theta \wedge \omega = 0. \tag{22}$$

Since $d\theta = F$, taking the interior product with some vector in (22) yields

$$F \wedge (U \lrcorner \omega) = -F(U) \wedge \omega, \qquad \forall U \in TM. \tag{23}$$

The rest of the proof is purely algebraic. Let X_i, A_j be local orthonormal basis of $H_1 := TM_1$ and $H_2 := TM_2$. By Lemma 4.6 we have

$$F = \sum_{i,j} f_{ij} X_i \wedge A_j,$$

therefore $F = \sum X_i \wedge F(X_i) = \sum A_j \wedge F(A_j)$. We also introduce the notations $\phi = \sum F(X_i) \wedge (X_i \lrcorner \omega) = -\sum A_j \wedge (F(A_j) \lrcorner \omega)$ and $\psi = \sum F(A_j) \wedge (A_j \lrcorner \omega) = -\sum X_i \wedge (F(X_i) \lrcorner \omega)$.

For every $\alpha \in p_1^*(\Lambda_0^p M_1) \otimes p_2^*(\Lambda_0^q M_2)$ we have $\sum X_i \wedge (X_i \lrcorner \alpha) = p\alpha$ and $\sum A_j \wedge (A_j \lrcorner \alpha) = q\alpha$. Taking the wedge product with X in (20), summing over $X = X_i$ and using (23) yields

$$0 = -k_1 F(A) \wedge \omega - F \wedge (A \lrcorner \omega) + A \wedge \psi = (1 - k_1) F(A) \wedge \omega + A \wedge \psi. \tag{24}$$

One last contraction with A_i in (24) gives $(k_1 + n_2 - k_2)\psi = 0$ (note that $\psi \in p_1^*(\Lambda_0^{k_1+1} M_1) \otimes p_2^*(\Lambda_0^{k_2-1} M_2)$). Plugging back into (24) yields

$$(1 - k_1) F(A) \wedge \omega = 0, \qquad \forall A \in TM_2. \tag{25}$$

In a similar way one obtains

$$(1 - k_2) F(X) \wedge \omega = 0, \qquad \forall X \in TM_1, \tag{26}$$

and, by replacing ω with its Hodge dual, and taking the Hodge dual of the equations for $*\omega$ analogous to (25) and (26), we also get

$$(n_1 - k_1 - 1) F(A) \lrcorner \omega = 0, \qquad \forall A \in TM_2, \tag{27}$$

$$(n_2 - k_2 - 1) F(X) \lrcorner \omega = 0, \qquad \forall X \in TM_1. \tag{28}$$

If $2 \le k_1 \le n_1 - 2$, equations (25) and (27) show that $F = 0$. Similarly, if $2 \le k_2 \le n_2 - 2$, equations (26) and (28) show that $F = 0$. We are thus left with four cases:

$$(k_1, k_2) \in \{(1,1), (1, n_2 - 1), (n_1 - 1, 1), (n_1 - 1, n_2 - 1)\}. \tag{29}$$

Now, the pull-back to M of the Hodge operator of M_1 defines an operator on $\Lambda_0^* M$, which maps each component $p_1^*(\Lambda_0^{k_1} M_1) \otimes p_2^*(\Lambda_0^{k_2} M_2)$ in the decomposition (19) isomorphically onto $p_1^*(\Lambda_0^{n_1 - k_1} M_1) \otimes p_2^*(\Lambda_0^{k_2} M_2)$ by

$$*_1[p_1^*(\omega_1) \wedge p_2^*(\omega_2)] := p_1^*(*\omega_1) \wedge p_2^*(\omega_2).$$

One defines $*_2$ in a similar way and Lemma 3.3 shows that $*_1$ and $*_2$ are D-parallel. Using these "partial Hodge operators", it suffices to study only the first case in (29), i.e. we can assume that ω is a D-parallel 2-form in $p_1^*(\Lambda_0^1 M_1) \otimes p_2^*(\Lambda_0^1 M_2)$.

As $n \ge 5$, by looking at the case of D-parallel 2-forms treated above, we see that D is closed unless ω is decomposable. But this would imply that $\omega = \eta_1 \wedge \eta_2$, where each of the weightless 1-forms η_i define a D-parallel distribution included in H_i. The forms η_i are thus D-parallel and Lemma 5.7 implies that D is closed.

□

Looking back to the proof of Theorem 5.8, we see how different the non-closed case is from the case of a closed Weyl structure, and this despite the fact that the results are essentially similar: as in the classical Riemannian case, a weightless form which is parallel for a Weyl structure either defines a special, irreducible, holonomy, or the manifold is locally a product and the form is a linear combination of pull-backs of the volume forms of the factors (in the Riemannian case, other pull-backs may occur if the factors have reduced holonomy).

But while in Riemannian geometry the richer case is the one with irreducible, non-generic, holonomy – and these *special geometries*, despite extensive research in the last decades, are far from being completely understood –, in non-closed Weyl geometry this situation occurs only in dimension 4, and there it defines a rather simple structure. It appears that for *non-closed* Weyl structures it is the case with *reduced* holonomy which is more interesting, and the reason is that the holonomy group – although defining a local product structure on the manifold – is not itself a product like in the Riemannian situation.

The following consequence of the Theorem 5.8 sheds some light on the reduced holonomy group of a non-closed Weyl structure:

Corollary 5.9. *Let $(M_1^{n_1}, c_1)$ and $(M_2^{n_2}, c_2)$ be two conformal manifolds and let c be a non-closed conformal product structure on $M = M_1 \times M_2$ with adapted Weyl structure D. If the dimension of M is different from 2 and 4, then the only D-parallel distributions on M are the kernels H_2 and H_1 of the canonical projections of M on M_1, respectively M_2.*

Proof. Let ω_i denote the (D-parallel) weightless volume form of H_i. If H is a D-parallel distribution, its weightless volume form ω is D-parallel. Since D is non-closed and $\dim(M) \neq 4$, Theorem 5.8 shows that ω is either proportional to ω_1 or ω_2 (in which case H is equal to H_1 or H_2), or $n_1 = n_2$ and $\omega = \lambda\omega_1 + \mu\omega_2$ for some $\lambda, \mu \in \mathbb{R}^*$. We claim that this last case is impossible. Let X be some vector field in H whose projections X_i onto H_i are both non-vanishing (such a vector field exists locally because H is not equal to H_1 or H_2), and let $\sigma = c(X, .)$ be the dual 1-form of weight 1, which decomposes correspondingly as $\sigma = \sigma_1 + \sigma_2$. We clearly have $\sigma \wedge \omega = 0$, whereas

$$\sigma \wedge (\lambda\omega_1 + \mu\omega_2) = \lambda\sigma_2 \wedge \omega_1 + \mu\sigma_1 \wedge \omega_2$$

and the two terms on the right hand side are non-vanishing and have bi-degree $(n_1, 1)$ and $(1, n_2)$ with respect to the decomposition (19). The assumption $n_1 + n_2 \neq 2$ shows that their sum can not vanish, a contradiction which proves our claim.

□

Remark 5.10. The restriction on the dimension of M is necessary. In dimension 2, any conformal product of two 1-dimensional manifolds admits parallel lines (distributions of rank 1) in any directions: simply consider linear combinations of the two volume forms on the factors. But not all such conformal products are closed. In dimension 4, there exist local examples of non-closed conformal products on $\mathbb{C}^2$, whose adapted Weyl structure preserves every complex line (see [6]).

A more detailed description of these low-dimensional cases, the study of the compact situation, as well as other issues concerning the holonomy of non-closed conformal products will make the object of a forthcoming paper.

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